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Supersymmetric quasi-Hermitian Hamiltonians with point interactions on a loop

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Abstract

We explore some aspects of $\mathcal{PT}$-symmetric Hamiltonians with two point interactions. We determine classes of point interactions for which the Hamiltonians are supersymmetric. We prove that these Hamiltonians are quasi-Hermitian and find a very simple formula for the metric operator Θ and its square root ϱ as well. Further, we present the quasi-Hermitian Hamiltonian (with one-point interaction) with a continuous spectrum.

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1. Introduction

Although quasi-Hermitian operators [1] are non-Hermitian they possess a real spectrum. $\mathcal{PT}$-symmetric Hamiltonians are not automatically quasi-Hermitian, however they often serve as a suitable starting point. In order to obtain not too technically complicated systems, we may consider $\mathcal{PT}$-symmetric point interactions instead of usual complex potentials. The classification of $\mathcal{PT}$-symmetric point interactions is provided by [2]. Particular examples of quasi-Hermitian models with one-point interaction may be found in [3] or [4]. The former presents the model with a closed form of metric operator Θ, while the latter offers an approximate formula.

The quasi-Hermitian operator may be mapped by similarity transformation $\varrho = \sqrt{\Theta}$ to the self-adjoint one [5]. However, an explicit calculation of $\sqrt{\Theta}$ may be very difficult even for a not very complicated metric operator. Recent discussion of the scattering theory [6] for non-Hermitian Hamiltonians is partly based on this similarity transformation as well. Thus an explicit example of ϱ may be very useful.

We present several Hamiltonians with $\mathcal{PT}$-symmetric point interactions which turn out to be quasi-Hermitian. We construct the metric operator Θ as well as its square root ϱ and we find a closed-form formula for both operators.

The first two models are Hamiltonians with two $\mathcal{PT}$-symmetric point interactions which are compatible with supersymmetric structure. The problem of finding suitable boundary conditions allowing supersymmetry has already been solved for the self-adjoint case in [7]. We explore a generalization to the $\mathcal{PT}$-symmetric systems.

A special type of boundary conditions found for supersymmetric models inspire us to consider the quasi-Hermitian Hamiltonian with continuous spectrum. Although the point spectrum of the Hamiltonian is empty, we found the metric operator and its square root.

We recall the definition of the quasi-Hermitian operator.

Definition 1. *Densely defined operator A acting on a Hilbert space $\mathcal{H}$ is called quasi-Hermitian, if there exists an operator Θ with properties*

(1) $\Theta \in B(\mathcal{H})$,
(2) $\Theta > 0$,
(3) $A^* = \Theta A \Theta^{-1}$.

It follows from the more mathematically oriented analysis on quasi-Hermiticity [8] that all requirements on metric operator are essential and is needed to verify them carefully, especially the domains of definition.

2. Supersymmetric models with point interactions

We consider a system on the finite interval $(-l, l)$ and two point interactions, at $x = 0$ and $x = \pm l$ (i.e. interaction at the origin and between the two end points). Every $\mathcal{PT}$-symmetric point interaction at $x = a$ may be described by boundary conditions ([2], theorem 2, connected case). In order to show connection with the self-adjoint case [7] we rewrite these conditions in the following form, using the same parameters $b \geqslant 0$, $c \geqslant -1/b$, $\theta, \phi \in [0, 2\pi)$ as in [2]:

$$(C - I)\Psi(a) + (C + I)\Psi'(a) = 0, \tag{1}$$

where

$$\Psi(a) = \begin{pmatrix} \psi(a+) \\ \psi(a-) \end{pmatrix}, \qquad \Psi'(a) = \begin{pmatrix} \psi'(a+) \\ -\psi'(a-) \end{pmatrix}, \tag{2}$$

$$C = \begin{pmatrix} \frac{(b-c)\,\mathrm{e}^{\mathrm{i}\phi} + \sqrt{1+bc}(\mathrm{e}^{2\mathrm{i}\phi}-1)}{(b+c)\,\mathrm{e}^{\mathrm{i}\phi} + \sqrt{1+bc}(\mathrm{e}^{2\mathrm{i}\phi}+1)} & \frac{2\mathrm{e}^{(\theta+\phi)}}{(b+c)\,\mathrm{e}^{\mathrm{i}\phi} + \sqrt{1+bc}(\mathrm{e}^{2\mathrm{i}\phi}+1)} \\ \frac{2\mathrm{e}^{(-\theta+\phi)}}{(b+c)\,\mathrm{e}^{\mathrm{i}\phi} + \sqrt{1+bc}(\mathrm{e}^{2\mathrm{i}\phi}+1)} & \frac{(b-c)\,\mathrm{e}^{\mathrm{i}\phi} - \sqrt{1+bc}(\mathrm{e}^{2\mathrm{i}\phi}-1)}{(b+c)\,\mathrm{e}^{\mathrm{i}\phi} + \sqrt{1+bc}(\mathrm{e}^{2\mathrm{i}\phi}+1)} \end{pmatrix}, \tag{3}$$

and the symbols $a\pm$ have the usual meaning of limits $\psi(a\pm) = \lim_{x\to a\pm} \psi(x)$.

Our aim is to find $\mathcal{PT}$-symmetric systems on a loop $(-l, l)$ with two point interactions (at 0 and l) which are supersymmetric. Regarding usual simplicity (exact solvability in terms of elementary functions) of Hamiltonians with point interactions and moreover its special structure (SUSY), we intend to find explicitly the spectra and metric Θ operators for these systems.

In order to obtain supersymmetric system with supercharges $Q_{1,2} \propto \frac{\mathrm{d}}{\mathrm{d}x}$,

$$\{Q_a, Q_b\} = H\delta_{ab}, \tag{4}$$

we have to restrict boundary conditions (1)–(3). We expect that we receive boundary conditions which connect values of functions and values of derivatives separately as well as in a self-adjoint case [7].

We recall briefly the procedure of finding suitable boundary conditions compatible with supersymmetry presented in [7] and we modify it to the $\mathcal{PT}$-symmetric case.

If φ is an eigenfunction of H, then $Q\varphi$ is also the eigenfunction of H corresponding to the same eigenvalue (or $Q\varphi = 0$). Since it is not guaranteed for general boundary conditions that $Q\varphi$ satisfies (1) although φ does, supercharges cannot be only derivatives multiplied by a scalar. We take an eigenfunction φ of H

$$H\varphi = E\varphi \tag{5}$$

and denote $\chi \equiv Q\varphi$. Since the supercharge is proportional to the derivative, boundary values of χ are related to those of φ':

$$\Psi_\chi(a) \equiv \begin{pmatrix} \chi(a+) \\ \chi(a-) \end{pmatrix} = M \begin{pmatrix} \varphi'(a+) \\ -\varphi'(a-) \end{pmatrix}, \tag{6}$$

where M is an invertible matrix. φ is an eigenfunction of H, hence φ'' is proportional to φ and

$$\Psi'_\chi(a) \equiv \begin{pmatrix} \chi'(a+) \\ -\chi'(a-) \end{pmatrix} = E\tilde{M} \begin{pmatrix} \varphi(a+) \\ \varphi(a-) \end{pmatrix}, \tag{7}$$

where $\tilde{M}$ is an invertible matrix again. When we combine (1), (6) and (7), we arrive at

$$(C - I)\tilde{M}^{-1}\Psi'_\chi(a) + E(C + I)M^{-1}\Psi_\chi(a) = 0. \tag{8}$$

Boundary conditions have to be energy independent and Ψ_χ, Ψ'_χ are not zero vectors simultaneously. Therefore, $(C \pm I)$ must be singular matrices, i.e. eigenvalues of C are ± 1. This constraint restricts the general form of C to two possibilities:

$$C_\pm = \pm \begin{pmatrix} \mathrm{i}\tan\phi & \frac{\mathrm{e}^{\mathrm{i}\theta}}{\cos\phi} \\ \frac{\mathrm{e}^{-\mathrm{i}\theta}}{\cos\phi} & -\mathrm{i}\tan\phi \end{pmatrix}, \tag{9}$$

i.e. parameters b, c are equal to zero, however, the range of θ, ϕ is preserved.

After reparametrization of $C_\pm$ elements using new both $\vec{\beta}$ and $\vec{b}$ parameters

$$\beta_1 = b_1 = -\frac{\cos\theta}{\cos\phi}, \qquad \beta_2 = b_2 = \frac{\sin\theta}{\cos\phi}, \qquad \beta_3 = \mathrm{i}b_3 = -\mathrm{i}\tan\phi, \tag{10}$$

$$(\vec{\beta})^2 = b_1^2 + b_2^2 - b_3^2 = 1, \qquad \beta_{1,2} \in \mathbb{R}, \qquad \beta_3 \in \mathrm{i}\mathbb{R}, \qquad b_{1,2,3} \in \mathbb{R}, \tag{11}$$

we arrive at

$$C_\pm = \exp\left(\mathrm{i}\frac{\pi}{2}(I \pm \vec{\beta}\cdot\vec{\sigma})\right) \tag{12}$$

where $\vec{\sigma}$ are the Pauli matrices.

We use parameters $\vec{\beta}$ in order to write following expressions in a more elegant way and to show connection with the self-adjoint case, where parameters real $\vec{\alpha}$ are used [7]. However, we move to the real parameters $\vec{b}$ in the following sections to avoid the tricky structure of $\vec{\beta}$, β_3 is not real.

We summarize results of [7] in the following (all technical details and derivations of formulae can be found there) and adapt the results to the $\mathcal{PT}$-symmetric case. Fortunately, the transition from the self-adjoint case turned out to be very easy, in fact a shift $\vec{\alpha} \mapsto \vec{\beta}$ is needed only. A direct connection can be found in relation (12) because it is a slight generalization of the standard one (16). Boundary conditions for the self-adjoint case are described by the unitary matrix U and the real parameter L_0 in equation

$$(U - I)\Psi(a) + \mathrm{i}L_0(U + I)\Psi'(a) = 0, \tag{13}$$

where Ψ, Ψ' were defined in (2). We may use an exponential form of unitary matrix

$$U \equiv U_g(\theta_+, \theta_-) = \exp\{\mathrm{i}\theta_+ P_g^+ + \mathrm{i}\theta_- P_g^-\}, \tag{14}$$

where $P_g^\pm$ are orthogonal projectors:

$$\begin{aligned} &P_g^\pm = \tfrac{1}{2}(I \pm g), \qquad g = \vec{\alpha}\cdot\vec{\sigma}, \qquad \vec{\alpha} \in \mathbb{R}^3, \qquad (\vec{\alpha})^2 = 1, \\ &\left(P_g^\pm\right)^2 = P_g^\pm = \left(P_g^\pm\right)^*, \qquad P_g^\pm P_g^\mp = 0, \qquad P_g^+ + P_g^- = I. \end{aligned} \tag{15}$$

Supersymmetry restricts (proof in [7]) these general conditions to

$$U_g(\pi, 0) = \exp\left\{\mathrm{i}\frac{\pi}{2}(I \pm \vec{\alpha}\cdot\vec{\sigma})\right\} \tag{16}$$

Although we cannot use the exponential form for the general matrix C (3), both matrices $U_g(\pi, 0)$ and $C_\pm$ with restricted parameters may be written in the exponential form (12), (16). We note that the only difference between $U_g(\pi, 0)$ and $C_\pm$ is the structure of $\vec{\alpha}$ and $\vec{\beta}$. This fact allows us to obtain the supercharges, eigenvalues and eigenfunctions of Hamiltonian very easily from the self-adjoint case.

In order to express boundary conditions in a more convenient way, we use operators $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ which have been already introduced in [7],

$$(\mathcal{P}\psi)(x) = \psi(-x), \qquad (\mathcal{R}\psi)(x) = (\vartheta(x) - \vartheta(-x))\psi(x), \qquad \mathcal{Q} = -\mathrm{i}\mathcal{R}\mathcal{P}, \tag{17}$$

where ϑ is a Heaviside step function. The operators are labeled in the following way:

$$\mathcal{P}_1 \equiv \mathcal{P}, \qquad \mathcal{P}_2 \equiv \mathcal{Q}, \qquad \mathcal{P}_3 \equiv \mathcal{R}. \tag{18}$$

The set of these operators forms an algebra of Pauli matrices, i.e.

$$[\mathcal{P}_l, \mathcal{P}_m] = 2\mathrm{i}\varepsilon_{lmn}\mathcal{P}_n, \qquad \{\mathcal{P}_l, \mathcal{P}_m\} = 2\delta_{lm} I. \tag{19}$$

Next, the operator $\mathcal{G}$ associated with $g = \vec{\beta}\cdot\vec{\sigma}$ is introduced,

$$\mathcal{G} = \vec{\beta}\cdot\vec{\mathcal{P}}, \tag{20}$$

obeying $\mathcal{G}^2 = I$, $\mathcal{G}^* \neq \mathcal{G}$, $[\mathcal{G}, \mathcal{P}\mathcal{T}] = 0$. (In the self-adjoint case, $\vec{\beta}$ is replaced by $\vec{\alpha}$ and $\mathcal{G}$ is self-adjoint [7].) It allows us to decompose any function ψ into two eigenfunctions of $\mathcal{G}$:

$$\psi_\pm = \tfrac{1}{2}(I \pm \mathcal{G})\psi, \qquad \psi = \psi_+ + \psi_-, \qquad \mathcal{G}\psi_\pm = \pm\psi_\pm. \tag{21}$$

Boundary conditions at $x = a$ corresponding to $C_\pm$ are now expressed in the form

$$\text{type}+: \psi_+(a+) = \psi_-'(a-) = 0, \qquad \text{type}-: \psi_+'(a+) = \psi_-(a-) = 0. \tag{22}$$

Hence we study two types of models: (++) and (+−). (++) denotes the interaction of the type + at $x = 0$ and of the type − at $x = l$ (at $x = l$ boundary conditions connect $x = -l$ and $x = l$). The other combinations provide equivalent models.

2.1. *Model of the type* (++)

We work in the Hilbert space $L^2(-l, l)$. The domain of definition of our Hamiltonian $H_1 \equiv H_{++} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ consists of functions $\psi \in \mathrm{AC}^2(\Omega)$ which obey boundary conditions (++) at $x = 0$ and $x = \pm l$. We adopted notation of [13], i.e. $\psi \in \mathrm{AC}^2(\Omega)$ if ψ, ψ' are absolutely continuous at Ω and $\psi'' \in L^2(-l, l)$. For our system, $\Omega = (-l, 0) \cup (0, l)$.

$$\begin{aligned} \mathrm{Dom}(H_1): \psi \in \mathrm{AC}^2(\Omega), \\ &(b_1 + \mathrm{i}b_2)\psi(0+) + (1 - \mathrm{i}b_3)\psi(0-) = 0, \\ &(b_1 + \mathrm{i}b_2)\psi'(0+) + (1 + \mathrm{i}b_3)\psi'(0-) = 0, \\ &(b_1 + \mathrm{i}b_2)\psi(l) + (1 - \mathrm{i}b_3)\psi(-l) = 0, \\ &(b_1 + \mathrm{i}b_2)\psi'(l) + (1 + \mathrm{i}b_3)\psi'(-l) = 0, \\ &b_{1,2,3} \in \mathbb{R}, \qquad b_1^2 + b_2^2 - b_3^2 = 1, \end{aligned} \tag{23}$$

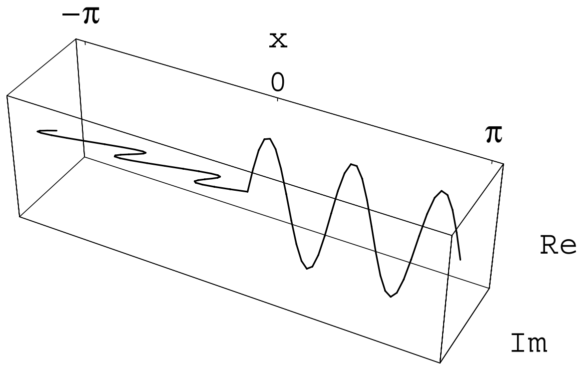

Figure 1. Eigenfunction ψ_{5+}, $\vec{b} = (10.000, 5.600, 11.417)$, $l = \pi$.

where $b_{1,2,3} \in \mathbb{R}$ and $b_1^2 + b_2^2 - b_3^2 = 1$. Since the fractions $(1 \pm \mathrm{i}b_3)/(b_1 + \mathrm{i}b_2)$ have absolute values equal to 1, boundary conditions may be rewritten as (similarly for $x = \pm l$)

$$\psi(0+) = \mathrm{e}^{\mathrm{i}\tau_1}\psi(0-), \qquad \psi'(0+) = \mathrm{e}^{\mathrm{i}\tau_2}\psi'(0-), \qquad \tau_{1,2} \in \mathbb{R}. \tag{24}$$

Parameters $\tau_{1,2}$ are different if $b_3 \neq 0$, the case $b_3 = 0$ corresponds to the self-adjoint setting.

It is not difficult to find the adjoint operator H_1^* directly from the definition of adjoint operator [13], i.e. using standard technique, integration by parts, etc.

$$\begin{aligned}
&\mathrm{Dom}(H_1^*) : \psi \in \mathrm{AC}^2(\Omega),\\
&\qquad (b_1 + \mathrm{i}b_2)\psi(0+) + (1 + \mathrm{i}b_3)\psi(0-) = 0,\\
&\qquad (b_1 + \mathrm{i}b_2)\psi'(0+) + (1 - \mathrm{i}b_3)\psi'(0-) = 0,\\
&\qquad (b_1 + \mathrm{i}b_2)\psi(l) + (1 + \mathrm{i}b_3)\psi(-l) = 0,\\
&\qquad (b_1 + \mathrm{i}b_2)\psi'(l) + (1 - \mathrm{i}b_3)\psi'(-l) = 0,\\
&\qquad b_{1,2,3} \in \mathbb{R}, \qquad b_1^2 + b_2^2 - b_3^2 = 1.
\end{aligned} \tag{25}$$

Since H_1 is equal to $H_{-b_3}^*$ (we change the sign of b_3 in (23) and take the adjoint) it is closed. We remark that H_1 is $\mathcal{P}$-pseudo-Hermitian only for $b_2 = 0$.

Eigenvalues of H_1 are the same as in the self-adjoint case [7], eigenfunctions differ only in the substitution $\vec{\alpha} \mapsto \vec{b}$, i.e.

$$\begin{aligned}
&E_n = \left(\frac{n\pi}{l}\right)^2,\\
&\psi_{n+}(x) = C_n\left(\vartheta(x) - \vartheta(-x)\frac{b_1 + \mathrm{i}b_2}{1 + \mathrm{i}b_3}\right)\sin\frac{n\pi}{l}x,\\
&\psi_{n-}(x) = C_n\left(\vartheta(x) - \vartheta(-x)\frac{b_1 + \mathrm{i}b_2}{1 - \mathrm{i}b_3}\right)\cos\frac{n\pi}{l}x, \qquad n \in \mathbb{N}_0,
\end{aligned} \tag{26}$$

where $\vartheta(x)$ is a Heaviside step function and C_n are normalization constants. Eigenfunctions of the Hamiltonian $\psi_{n\pm}$ are eigenfunctions of operator $\mathcal{G}$ (20) as well, corresponding to the eigenvalues ± 1 (the generalization of the proof from the self-adjoint case [7] is straightforward). Figures 1 and 2 illustrate eigenfunctions $\psi_{s\pm}$, point interactions at the origin and end points rotate wavefunction in the complex plane.

Energy levels are doubly degenerate except the lowest one as we expected for the supersymmetric system. Supercharges $Q_{1,2}$ may be obtained from the self-adjoint case [7]

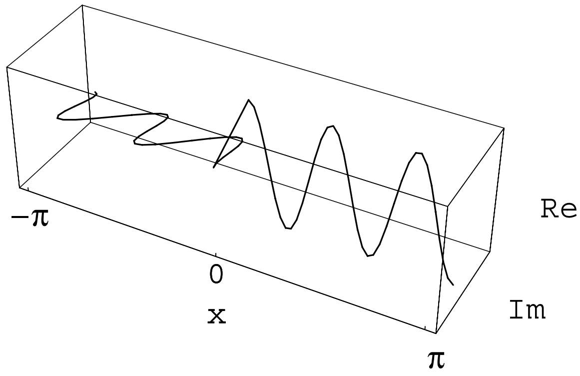

Figure 2. Eigenfunction ψ_{5-}, $\vec{b} = (10.000, 5.600, 11.417)$, $l = \pi$.

easily again, by substitution $\vec{\alpha} \mapsto \vec{\beta}$

$$\mathcal{Q}_{1,2} = \mathrm{i}\frac{\sqrt{2}}{2}\mathcal{G}_{1,2}\mathcal{P}_3\frac{\mathrm{d}}{\mathrm{d}x}, \tag{27}$$

where

$$\mathcal{G}_{1,2} = \vec{\gamma}_{1,2}\cdot\vec{\mathcal{P}}, \qquad (\vec{\gamma}_{1,2})^2 = 1 \qquad \text{and} \qquad \vec{\gamma}_{1,2}\cdot\vec{\beta} = \vec{\gamma}_1\cdot\vec{\gamma}_2 = 0. \tag{28}$$

Eigenfunctions of Hamiltonian have a very simple form and this fact allows us to construct the metric Θ operator using the idea of Mostafazadeh [9, 10]

$$\Theta = \sum_n c_n\langle\phi_n,\cdot\rangle\phi_n, \tag{29}$$

where c_n are positive numbers and ϕ_n are eigenfunctions of H_1^*. An alternative approach of Krejčiřík [11] using spectral theorem can be applied as well. We denote $\phi_{n\pm}$ eigenfunctions of H^* and normalize them in a special way:

$$\begin{aligned}
\phi_{n+}(x) &= \sqrt{\frac{2}{l}}\left(\vartheta(x) - \vartheta(-x)\frac{b_1+\mathrm{i}b_2}{1-\mathrm{i}b_3}\right)\sin\frac{n\pi}{l}x,\\
\phi_{0-}(x) &= \frac{1}{\sqrt{l}}\left(\vartheta(x) - \vartheta(-x)\frac{b_1+\mathrm{i}b_2}{1+\mathrm{i}b_3}\right),\\
\phi_{n-}(x) &= \sqrt{\frac{2}{l}}\left(\vartheta(x) - \vartheta(-x)\frac{b_1+\mathrm{i}b_2}{1+\mathrm{i}b_3}\right)\cos\frac{n\pi}{l}x.
\end{aligned} \tag{30}$$

Sets $\{e_n^\pm\}_{n=1}^\infty$, $\{f_n^\pm\}_{n=0}^\infty$,

$$e_n^\pm(x) = \sqrt{\frac{2}{l}}\vartheta(\pm x)\sin\frac{n\pi}{l}x, \tag{31}$$

$$f_0^\pm(x) = \sqrt{\frac{1}{l}}\vartheta(\pm x), \qquad f_n^\pm(x) = \sqrt{\frac{2}{l}}\vartheta(\pm x)\cos\frac{n\pi}{l}x, \tag{32}$$

form orthonormal bases of $L^2(-l,0)$ and $L^2(0,l)$. We express $\phi_{n\pm}$ in terms of $e_n^\pm$, $f_n^\pm$ and calculate

$$\begin{aligned}
\langle\phi_{n+},\psi\rangle\phi_{n+} &= \langle e_n^+,\psi\rangle e_n^+ + \langle e_n^-,\psi\rangle e_n^- + \frac{b_1+\mathrm{i}b_2}{1-\mathrm{i}b_3}\langle e_n^-,\mathcal{P}\psi\rangle e_n^- + \frac{b_1-\mathrm{i}b_2}{1+\mathrm{i}b_3}\langle e_n^+,\mathcal{P}\psi\rangle e_n^+,\\
\langle\phi_{n-},\psi\rangle\phi_{n-} &= \langle e_f^+,\psi\rangle f_n^+ + \langle f_n^-,\psi\rangle f_n^- - \frac{b_1+\mathrm{i}b_2}{1+\mathrm{i}b_3}\langle f_n^-,\mathcal{P}\psi\rangle f_n^- - \frac{b_1-\mathrm{i}b_2}{1-\mathrm{i}b_3}\langle f_n^+,\mathcal{P}\psi\rangle f_n^+.
\end{aligned} \tag{33}$$

In order to find a metric Θ, we calculate the sum (29)

$$\begin{aligned}\Theta &= \operatorname*{s\text{--}lim}_{N\to\infty} \frac{1}{2}\left(\sum_{n=1}^{N} \langle \phi_{n+}, \cdot\rangle \phi_{n+} + \sum_{n=0}^{N} \langle \phi_{n-}, \cdot\rangle \phi_{n-}\right)\\ &= I - \frac{\mathrm{i}b_3}{b_1+\mathrm{i}b_2} P^+\mathcal{P} + \frac{\mathrm{i}b_3}{b_1-\mathrm{i}b_2} P^-\mathcal{P},\end{aligned} \tag{34}$$

where $\mathcal{P}$ is a parity and $P^{\pm}$ are orthogonal projectors

$$(P^{\pm}\psi)(x) = \vartheta(\pm x)\psi(x), \qquad (P^{\pm})^2 = P^{\pm} = (P^{\pm})^*, \qquad P^+P^- = P^-P^+ = 0. \tag{35}$$

We prove that Θ fulfils all requirements of the definition of the quasi-Hermitian operator (1) in subsection 2.3.

2.2. *Model of the type* (+−)

The domain of definition of the Hamiltonian $H_2 \equiv H_{+-}$ reads

$$\begin{aligned}\mathrm{Dom}(H_2) : \psi \in \mathrm{AC}^2(\Omega),&\\ &(b_1+\mathrm{i}b_2)\psi(0+) + (1-\mathrm{i}b_3)\psi(0-) = 0,\\ &(b_1+\mathrm{i}b_2)\psi'(0+) + (1+\mathrm{i}b_3)\psi'(0-) = 0,\\ &(b_1+\mathrm{i}b_2)\psi(l) - (1+\mathrm{i}b_3)\psi(-l) = 0,\\ &(b_1+\mathrm{i}b_2)\psi'(l) - (1-\mathrm{i}b_3)\psi'(-l) = 0,\\ &b_{1,2,3} \in \mathbb{R}, \qquad b_1^2+b_2^2-b_3^2 = 1,\end{aligned} \tag{36}$$

eigenvalues and eigenfunctions are

$$\begin{aligned}E_n &= \left(\frac{(2n-1)\pi}{2l}\right)^2,\\ \psi_{n+}(x) &= C_n\left(\vartheta(x) - \vartheta(-x)\frac{b_1+\mathrm{i}b_2}{1+\mathrm{i}b_3}\right)\sin\frac{(n-1)\pi}{2l}x,\\ \psi_{n-}(x) &= C_n\left(\vartheta(x) - \vartheta(-x)\frac{b_1+\mathrm{i}b_2}{1-\mathrm{i}b_3}\right)\cos\frac{(n-1)\pi}{2l}x, \qquad n \in \mathbb{N}.\end{aligned} \tag{37}$$

Supercharges have exactly the same form as in the previous case (27), however the supersymmetric structure of this model is different because the zero energy level is absent.

We use analogous procedure to obtain metric operator. We express eigenfunctions of H_2^* in terms of e_n, f_n:

$$\begin{aligned}&e_0(x) = \frac{1}{\sqrt{2l}}, \qquad e_{2k-1}(x) = \frac{1}{\sqrt{l}}\sin\frac{(2k-1)\pi}{2l}x, \qquad e_{2k}(x) = \frac{1}{\sqrt{l}}\cos\frac{k\pi}{l}x,\\ &f_{2k-1}(x) = \frac{1}{\sqrt{l}}\cos\frac{(2k-1)\pi}{2l}x, \qquad f_{2k}(x) = \frac{1}{\sqrt{l}}\sin\frac{k\pi}{l}x,\end{aligned} \tag{38}$$

where sets $\{e_n\}_{n=0}^{\infty}$, $\{f_n\}_{n=1}^{\infty}$ form orthonormal bases of $L^2(-l,l)$. Summation (29) in the strong limit sense yields

$$\begin{aligned}\Theta = P^+(O_1+O_2)P^+ &+ P^-(O_1+O_2)P^- - \frac{b_1-\mathrm{i}b_2}{1+\mathrm{i}b_3}P^+O_1P^-\\ &-\frac{b_1+\mathrm{i}b_2}{1-\mathrm{i}b_3}P^-O_1P^+ - \frac{b_1-\mathrm{i}b_2}{1-\mathrm{i}b_3}P^+O_2P^- - \frac{b_1+\mathrm{i}b_2}{1+\mathrm{i}b_3}P^-O_2P^+,\end{aligned} \tag{39}$$

where $O_{1,2}$ are orthogonal projectors:

$$\begin{aligned} O_1 e_{2k} &= 0, & O_1 e_{2k-1} &= e_{2k-1}, \\ O_2 f_{2k} &= 0, & O_2 f_{2k-1} &= f_{2k-1}. \end{aligned} \tag{40}$$

This result is derived directly from sum (29), nevertheless operators O_1 and O_2 are projectors, respectively, on the odd and even parts of the function, i.e.

$$O_1 = \tfrac{1}{2}(I - \mathcal{P}), \qquad O_2 = \tfrac{1}{2}(I + \mathcal{P}). \tag{41}$$

Hence the metric operator Θ has exactly the same form as in the previous case (34).

2.3. Metric operator

Theorem 2. *Operators $H_{1,2}$ are quasi-Hermitian, the metric operator Θ reads*

$$\Theta = I - \frac{\mathrm{i}b_3}{b_1 + \mathrm{i}b_2} P^+\mathcal{P} + \frac{\mathrm{i}b_3}{b_1 - \mathrm{i}b_2} P^-\mathcal{P}, \tag{42}$$

where $b_{1,2,3} \in \mathbb{R}$, $b_1^2 + b_2^2 - b_3^2 = 1$, $\mathcal{P}$ is a parity and $P^{\pm}$ are orthogonal projectors:

$$(\mathcal{P}\psi)(x) = \psi(-x), \qquad (P^{\pm}\psi)(x) = \vartheta(\pm x)\psi(x), \qquad (P^{\pm})^2 = P^{\pm} = (P^{\pm})^*. \tag{43}$$

Proof. We prove that Θ meets all requirements according to definition 1 in several steps:

(1) It is obvious that Θ is bounded, $\|\Theta\| \leqslant 3$.

(2) Θ is self-adjoint. Relation

$$\Theta^* = I - \frac{-\mathrm{i}b_3}{b_1 - \mathrm{i}b_2}\mathcal{P}P^+ + \frac{-\mathrm{i}b_3}{b_1 + \mathrm{i}b_2}\mathcal{P}P^-, \tag{44}$$

together with $\mathcal{P}P^{\pm} = P^{\mp}\mathcal{P}$, yields the result.

(3) Θ is positive:

$$\begin{aligned} \langle\psi, \Theta\psi\rangle &= \|\psi\|^2 - \frac{\mathrm{i}b_3}{b_1 + \mathrm{i}b_2} J + \frac{\mathrm{i}b_3}{b_1 - \mathrm{i}b_2}\overline{J} \\ &\geqslant \|\psi\|^2 - \left|\frac{\mathrm{i}b_3}{b_1 + \mathrm{i}b_2}\right| \left|J\right| \left|1 - \frac{b_1 + \mathrm{i}b_2}{b_1 - \mathrm{i}b_2}\frac{\overline{J}}{J}\right|, \end{aligned} \tag{45}$$

where

$$J = \int_0^l \overline{\psi(x)}\psi(-x)\,\mathrm{d}x, \tag{46}$$

$$\begin{aligned} |J| &\leqslant \int_0^l |\psi(x)||\psi(-x)|\mathrm{d}x \leqslant \frac{1}{2}\int_0^l |\psi(x)|^2 + |\psi(-x)|^2\,\mathrm{d}x \\ &\leqslant \frac{1}{2}\left(\|P^+\psi\|^2 + \|P^-\psi\|^2\right) \leqslant \frac{1}{2}\|\psi\|^2. \end{aligned} \tag{47}$$

These estimates yield all together

$$\langle\psi, \Theta\psi\rangle \geqslant \underbrace{\left(1 - \frac{|b_3|}{\sqrt{1 + |b_3|^2}}\right)}_{c_0} \|\psi\|^2 \geqslant 0. \tag{48}$$

(4) Θ is invertible and $\Theta^{-1} \in B(\mathcal{H})$. Since $c_0 > 0$, inequality (48) yields $0 \notin \sigma(\Theta)$.

(5) Θ maps the domains of definition of $H_{1,2}$ and $H_{1,2}^*$ correctly, i.e. $\Theta\,\mathrm{Dom}(H_{1,2}) = \mathrm{Dom}(H_{1,2}^*)$. It is straightforward to calculate limits (for $\pm l$ analogously)

$$\begin{aligned}(\Theta\psi)(0+) &= \psi(0+) - \frac{\mathrm{i}b_3}{b_1+\mathrm{i}b_2}\psi(0-),\\ (\Theta\psi)(0-) &= \psi(0-) + \frac{\mathrm{i}b_3}{b_1-\mathrm{i}b_2}\psi(0+),\end{aligned} \tag{49}$$

and verify that $\Theta\psi$ satisfies boundary conditions of $\mathrm{Dom}(H_{1,2}^*)$. □

Proposition 3. *Operator ϱ is bounded, positive and $\varrho^2 = \Theta$,*

$$\varrho = a_1 I + a_2 P^+\mathcal{P} + \overline{a_2}P^-\mathcal{P}, \tag{50}$$

where

$$a_1 > 0, \qquad a_1^2 = \frac{1}{2}(1+\sqrt{1-|k|^2}), \qquad a_2 = \frac{k}{2a_1}, \qquad k = -\frac{\mathrm{i}b_3}{b_1+\mathrm{i}b_2}. \tag{51}$$

Proof. It is clear that ϱ is bounded:

$$\varrho^2 = \left(a_1^2 + |a_2|^2\right)I + 2a_1a_2P^+\mathcal{P} + 2a_1\overline{a_2}P^-\mathcal{P}, \tag{52}$$

where we used identities $\mathcal{P}P^\pm = P^\mp\mathcal{P}$ and $P^+ + P^- = I$. Slightly modified estimations (45)–(47) yield

$$\langle\psi, \varrho\psi\rangle \geqslant (a_1 - |a_2|)\|\psi\|^2, \tag{53}$$

and $a_1 - |a_2| > 0$, whence ϱ is positive. □

2.4. Model with continuous spectrum

A motivation for the quasi-Hermitian model with continuous spectrum is a system on $L^2(-l,l)$, $H_3 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$:

$$\begin{aligned}\mathrm{Dom}(H_3) : \psi \in \mathrm{AC}^2(\Omega),&\\ &(b_1+\mathrm{i}b_2)\psi(0+) + (1-\mathrm{i}b_3)\psi(0-) = 0,\\ &(b_1+\mathrm{i}b_2)\psi'(0+) + (1+\mathrm{i}b_3)\psi'(0-) = 0,\\ &\psi(-l) = \psi(l) = 0,\\ &b_{1,2,3}\in\mathbb{R}, \qquad b_1^2+b_2^2-b_3^2 = 1.\end{aligned} \tag{54}$$

The eigenvalues and eigenfunctions of Hamiltonian read

$$\begin{aligned}E_n &= \left(\frac{n\pi}{2l}\right)^2,\\ \psi_{2n}(x) &= C_{2n-1}\left(\vartheta(x) - \vartheta(-x)\frac{b_1+\mathrm{i}b_2}{1+\mathrm{i}b_3}\right)\sin\frac{n\pi}{l}x,\\ \psi_{2n+1}(x) &= C_{2n}\left(\vartheta(x) - \vartheta(-x)\frac{b_1+\mathrm{i}b_2}{1-\mathrm{i}b_3}\right)\cos\frac{(2n+1)\pi}{2l}x, n\in\mathbb{N}_0.\end{aligned} \tag{55}$$

It is possible to show by using the same procedure as in the (++) and (+−) models that the strong limit of sum (29) is equal to the Θ (42) and all the properties of metric operator for H_3 are satisfied as well. Since metric operator does not depend on a scale parameter l explicitly, we may ask if the constructed operator Θ is a suitable metric for the system on a whole line. Indeed, we consider $L^2(\mathbb{R})$ and $H_4 = -\frac{d^2}{dx^2}$:

$$\begin{aligned} \mathrm{Dom}(H_4) : \psi \in \mathrm{AC}^2(\mathbb{R} - \{0\}), & \\ (b_1 + \mathrm{i}b_2)\psi(0+) + (1 - \mathrm{i}b_3)\psi(0-) = 0, & \\ (b_1 + \mathrm{i}b_2)\psi'(0+) + (1 + \mathrm{i}b_3)\psi'(0-) = 0, & \\ b_{1,2,3} \in \mathbb{R}, \qquad b_1^2 + b_2^2 - b_3^2 = 1. & \end{aligned} \tag{56}$$

We take the operator Θ (42) and extend it on $L^2(\mathbb{R})$ in a natural way. It is obvious that Θ maps correctly the domains (exactly the same proof as before) so that equality

$$H_4^* = \Theta H_4 \Theta^{-1} \tag{57}$$

holds. The proof of positivity stated in theorem 2 is valid for $C_0^\infty(\mathbb{R})$ functions, hence the positivity of extended Θ follows immediately from the density of $C_0^\infty(\mathbb{R})$ and boundedness of Θ. All these facts together show that H_4 is quasi-Hermitian operator (definition 1).

The spectrum of H_4 is different from the other models, it consists of $[0, \infty)$ of continuous spectrum and at most two eigenvalues which are real negative (theorem 2, [2]). For the presented model, the solution of $H_4\psi = E\psi$ for $E < 0$ satisfying both boundary conditions (56) and requirement of being in $L^2(\mathbb{R})$ is only a zero function. Therefore, point spectrum of H_4 is empty and $\sigma(H_4) = \sigma_c(H_4) = [0, \infty)$. Although no eigenfunctions are available (hence we cannot try to find a metric with the help of sum (29)), we showed that H_4 is quasi-Hermitian according to definition 1.

3. Concluding remarks

The requirement of supersymmetry in the system with $\mathcal{PT}$-symmetric points interactions restricts very strongly the choice of boundary conditions. The Hamiltonians are quasi-Hermitian and their spectrum is identical with the self-adjoint case. On the other hand, the simplicity of eigenfunctions allowed us to find a formula for both metric operator and its square root. Point interaction described by (23) seems to be the most elementary one, they do not combine the function and derivative. Example [3] shows that already special mixed boundary conditions lead to the more complicated systems. We remark again that it is possible to rewrite the boundary conditions in the following way:

$$\psi(0+) = \mathrm{e}^{\mathrm{i}\tau_1}\psi(0-), \qquad \psi'(0+) = \mathrm{e}^{\mathrm{i}\tau_2}\psi'(0-), \qquad \tau_1 \neq \tau_2 \tag{58}$$

which strongly resemble one particular class of self-adjoint extensions [13]

$$\psi(0+) = \mathrm{e}^{\mathrm{i}\tau}\psi(0-), \qquad \psi'(0+) = \mathrm{e}^{\mathrm{i}\tau}\psi'(0-). \tag{59}$$

Possible generalizations of the presented models lie in an increase in the number of interactions. Supersymmetric structure of the model with n interactions may be very interesting and simplicity of eigenfunctions allows us probably to find simple form of metric operator. Another generalization is inserting the boundary conditions to the two-dimensional model, analogously to [12].

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